On the weak Freese-Nation property of complete Boolean algebras

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This is still a preliminary version.
Any comments are appreciated.
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February 1, 2008

Abstract

The following results are proved:

(a) *In a model obtained by adding* $\aleph_2$ *Cohen reals, there is always a c.c.c. complete Boolean algebra without the weak Freese-Nation property.*

(b) *Modulo the consistency strength of a supercompact cardinal, the existence of a c.c.c. complete Boolean algebra without the weak Freese-Nation property is consistent with* GCH.

(c) *If a weak form of* $\square_\mu$ *and* $\mathrm{cof}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ *hold for eavch* $\mu > \mathrm{cf}(\mu) = \omega$*, then the weak Freese-Nation property of* $(\mathcal{P}(\omega), \subseteq)$ *is equivalent to the weak Freese-Nation property of any of* $\mathbb{C}(\kappa)$ *or* $\mathbb{R}(\kappa)$ *for uncountable* κ*.*

(d) *Modulo consistency of* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$*, it is consistent with* GCH *that* $\mathbb{C}(\aleph_\omega)$ *does not have the weak Freese-Nation property and hence the assertion in (c) does not hold, and also that adding* $\aleph_\omega$ *Cohen reals destroys the weak Freese-Nation property of* $(\mathcal{P}(\omega), \subseteq)$*.*

These results solve all of the problems listed in Fuchino-Soukup [5] and some other problems posed by S. Geschke.

*The first author was partially supported by Grant-in-Aid for Scientific Research (C) No. 10640099 of the Ministry of Education, Science, Sports and Culture, Japan.

Some of the results here, in particular earlier versions of the results in Section 5, were included in the second author's Ph.D. thesis [7].

This paper is [FGShS:712] of the third author's publications list. His research is supported by "The Israel Science Foundation".

The fourth author was partially supported by Grant-in-Aid for JSPS Fellows No. 98259 of the Ministry of Education, Science, Sports and Culture, Japan, and by Hungarian National

1 Introduction

A quasi-ordering $(P, \leq)$ is said to have *the weak Freese-Nation property* if there is a mapping $f : P \to [P]^{\leq \aleph_0}$ such that:

For any p, $q \in P$ with $p \leq q$ there is $r \in f(p) \cap f(q)$ such that $p \leq r \leq q$.

A mapping f as above is called a *weak Freese-Nation mapping* on P.

The weak Freese-Nation property was introduced in Chapter 4 of [8] as a weakening of a notion of almost freeness of Boolean algebras. The property was further studied in [4] and [5].

In [4], it is shown that $(\mathcal{P}(\omega_1), \subseteq)$ does not have the weak Freese-Nation property. If a complete Boolean algebra B does not have the c.c.c., then $(\mathcal{P}(\omega_1), \subseteq)$ can be completely embedded into B. Hence, in this case, B can not have the weak Freese-Nation property.

It is easily seen that every quasi-ordering of cardinality $\leq \aleph_1$ has the weak Freese-Nation property (see e.g. [4]). It follows that, under CH, $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property.

To simplify the formulation of some of the results below, let us say that a model of set-theory is *neat* if $\Box_\mu$ holds — what is actually needed in the following is merely a very weak variant of $\Box_\mu$ introduced in [5] (see before Proposition 4.3) — and $\mathrm{cof}([\mu]^\omega, \subseteq) = \mu^+$ for each $\mu > \mathrm{cf}(\mu) = \omega$.

In [4] and on [5], it was shown that if CH holds, then every c.c.c. complete Boolean algebra of size $< \aleph_\omega$ has the weak Freese-Nation property; and in a neat model, CH implies that every c.c.c. complete Boolean algebra has the weak Freese-Nation property. However, the following questions remained unanswered in [5]:

Question 1 ([5, Problem 5]) *Are the following equivalent?*
(i) $(\mathcal{P}(\omega), \subseteq)$ *has the weak Freese-Nation property;*
(ii) *every c.c.c. complete Boolean algebra has the weak Freese-Nation property.*

Question 2 ([5, Problem 2]) *Does* ZFC + GCH *imply that every c.c.c. complete Boolean algebra has the weak Freese-Nation property?*

Foundation for Scientific Research grant no. 25745.

Section 6 of the present paper was worked out during the conference "Algebra and Discrete Mathematics" in Hattingen, Germany 1999 which was organized by the third author and attended by all of the other authors.

The final version of the paper was then prepared during the Workshop on Set-Theoretical Topology 1999 at Erdős Center, Budapest Hungary 1999.

We give negative answers here: see Corollary 3.4 for question 1 and Theorem 4.2 for question 2. By the result in [5] already mentioned above, we need consistency strength of some large cardinal to give a negative answer for Question 2. Indeed, the ground model V in the negative solution of this problem is obtained by starting from a model of ZFC with a supercompact cardinal.

In [4] it was shown that if CH holds, then adding less than $\aleph_\omega$ many Cohen reals preserves the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$. By [5], in the generic extension obtained by adding any number of Cohen reals to a neat model satisfying CH, not only $(\mathcal{P}(\omega), \subseteq)$ but every tame c.c.c. complete Boolean algebra has the weak Freese-Nation property. Here, letting $P = \mathrm{Fn}(\tau, 2)$ (= the standard p.o. for adding τ Cohen reals), a Boolean algebra in a P-generic extension is said to be *tame*, if there is a P-name $\dot{\leq}$ of partial ordering of B and a mapping $t : B \to [\tau]^{\aleph_0}$ in V such that, for every $p \in P$ and $x, y \in B$, if $p \Vdash_P$ "$x \mathrel{\dot{\leq}} y$", then $p\restriction (t(x) \cup t(y)) \Vdash_P$ "$x \mathrel{\dot{\leq}} y$" (we assume here without loss of generality that B is chosen so that its underlying set is a ground model set).

These results suggest the following questions posed in [5]:

Question 3 ([5, Problem 3]) *Assume that $V[G]$ is a model obtained by adding Cohen reals to a model of* ZFC + CH. *Is it true that $\mathcal{P}(\omega)$ has the weak Freese-Nation property in $V[G]$?*

Question 4 ([5, Problem 4]) *Assume that $V[G]$ is a model obtained by adding $\aleph_2$ Cohen reals to a model of* ZFC + CH. *Is it true that every c.c.c. complete Boolean algebra (not just the tame ones) has the weak Freese-Nation property in $V[G]$?*

The results of the present paper answer these questions in the negative: see Theorem 6.1 for question 3 and Corollary 3.3 for question 4.

By the result in [5] already mentioned above, we need consistency strength of some large cardinal to give a negative solution of Question 3. Indeed, the ground model V in the negative solution of this problem given in Theorem 6.1 is obtained by starting from a model of ZFC with a large cardinal slightly stronger than a huge cardinal.

After the negative solution of Problem 5, the following question still remains:

Problem 1 *For which Boolean algebra B, the weak Freese-Nation property of B is equivalent with the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$?*

The following easy lemma is already a result in this direction.

Lemma 1.1 *The following are equivalent:*

(a) $(\mathcal{P}(\omega), \subseteq)$ *has the weak Freese-Nation property;*
(b) $(\mathcal{P}(\omega), \subseteq^*)$ *has the weak Freese-Nation property;*
(c) $(\mathcal{P}(\omega)/fin, \subseteq^*)$ *has the weak Freese-Nation property;*
(d) $({}^\omega\omega, \leq)$ *has the weak Freese-Nation property;*
(e) $({}^\omega\omega, \leq^*)$ *has the weak Freese-Nation property.*
(f) $({}^\omega\mathbb{R}, \leq)$ *has the weak Freese-Nation property.* S.F.) ❑

Koppelberg [10] pointed out that the weak Freese-Nation property of Cohen algebra $\mathbb{C}(\omega)$ is equivalent to the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$. In the present paper, we show that it is also equivalent to the weak Freese-Nation property of the measure algebra $\mathbb{R}(\omega)$ (Proposition 5.1) and more over, in a mild model also with weak Freese-Nation property of $\mathbb{C}(\kappa)$ and/or $\mathbb{R}(\kappa)$ for any $\kappa \geq \aleph_0$ (Corollary 5.4). Here, we denote with $\mathbb{C}(\kappa)$ and $\mathbb{R}(\kappa)$ the c.c.c. complete Boolean algebras $Borel({}^\kappa 2)/meager({}^\kappa 2)$ and $Borel({}^\kappa 2)/null({}^\kappa 2)$ respectively. We show that some extra set-theoretic assumption are really necessary in Corollary 5.4 by constructing a model of GCH and the negation of weak Freese-Nation property for $\mathbb{C}(\aleph_\omega)$ starting from a model of GCH and Chang's conjecture for $\aleph_\omega$.

Assume that $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ is a finite support iteration such that forcing with $\dot{Q}_\alpha$ just adds a real to V^{P_α}. Then, as S. Geschke proved in [7] , if this iteration preserves the weak Freese-Nation property of $\mathcal{P}(\omega)$ then for all but ω_1 many α the partially ordered set $\dot{Q}_\alpha$ just adds one Cohen real. But by Corollary 3.3, in any model obtained by adding $\geq \aleph_2$ Cohen reals, there is a c.c.c. complete Boolean algebra B without the weak Freese-Nation property. So there is no easy way to blow up the continuum and to preserve the weak Freese-Nation property of all c.c.c complete Boolean algebras. Thus the following question seems to be a reasonable one:

Problem 2 *Does* CH *follow from the assumption that every c.c.c. complete Boolean algebra has the weak Freese-Nation property?*

If $\mathbf{b} > \aleph_1$ or if there is an $\aleph_2$-Luzin-gap, then $(\mathcal{P}(\omega), \subseteq)$ does not have the weak Freese-Nation property (see [4] and [5]). The following question ((\[5, Problem 1])) was raised against this background:

Suppose that $\mathcal{P}(\omega)$ does not have any increasing chain of length $\geq \omega_2$ with respect to $\subseteq^*$ and that there is no $\aleph_2$-Luzin gap. Does it follow that $\mathcal{P}(\omega)$ has the weak Freese-Nation property?

This problem can be solved negatively using results from [2] and [7]: Let V be a model of CH and $V[G]$ its generic extension by adding many random reals side

by side. Using results from [2] we see that in $V[G]$, there are neither increasing ω_2 chain in $\mathcal{P}(\omega)$ with respect to $\subseteq^*$ nor $\aleph_2$-Luzin gap. On the other hand S. Geschke [7] showed that in $V[G]$ $(\mathcal{P}(\omega), \subseteq)$ does not have the weak Freese-Nation property.

Consequences of the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ were studied in [10] and [6]. In the latter paper it was shown that a set-theoretic universe with the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ looks quite similar to a Cohen model. In particular, under the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$, all cardinal invariants which appear in [1] take the same value as in a Cohen model with the same size of $2^{\aleph_0}$.

Problem 3 *Find a combinatorial* ($\mathbf{\Pi}^1_1$) *characterization of weak Freese-Nation property of* $\mathcal{P}(\omega)$.

The weak Freese-Nation property of a quasi-ordering $(P, \leq)$ is actually a property of the corresponding partial ordering $(\overline{P}, \leq)$ obtained as the quotient structure of $(P, \leq)$ with respect to the equivalence relation "$x \leq y \land y \leq x$": $(P, \leq)$ has the weak Freese-Nation property if and only if $(\overline{P}, \leq)$ does.

The following criteria of the weak Freese-Nation property are used in the later sections. A partial ordering Q is said to be a *retract* of a partial ordering P if there are order preserving mappings $i : Q \to P$ and $j : P \to Q$ such that $j \circ i = id_Q$. Note that if P and Q are complete Boolean algebras and there is a strictly order-preserving embedding f of Q into P (i.e. f preserves ordering and incomparability) then we can always find order preserving $g : P \to Q$ such that $g \circ f = id_Q$: simply define g by $g(p) = \sum\{q \in Q : f(q) \leq p\}$ for $p \in P$.

Q is said to be a *σ-subordering of* P (notation: $Q \leq_\sigma P$) if, for every $p \in P$, $Q\restriction p = \{q \in Q : q \leq p\}$ has a countable cofinal subset and $Q \uparrow p = \{q \in Q : q \geq p\}$ has a countable coinitial subset. Note that if C is a complete subalgebra of a complete Boolean algebra B (notation: $C \leq_c B$) or a countable union of complete subalgebras of B, then it follows that $C \leq_\sigma B$.

Proposition 1.2 (a) (Lemma 2.7 in [4]) *If Q is a retract of P and P has the weak Freese-Nation property then Q has the weak Freese-Nation property.*

(b) (Lemma 2.3 (a) in [4]) *If $Q \leq_\sigma P$ and P has the weak Freese-Nation property, then Q also has the weak Freese-Nation property.*

(c) (Lemma 2.6 in [4]) *If P_α, $\alpha < \delta$ is an increasing sequence of partial orderings with the weak Freese-Nation property such that $P_\alpha \leq_\sigma P_{\alpha+1}$ for every $\alpha < \delta$ and $P_\gamma = \bigcup_{\alpha<\gamma} P_\alpha$ for all $\gamma < \delta$ with $\mathrm{cf}(\gamma) > \omega$, then $P = \bigcup_{\alpha<\delta} P_\alpha$ also has the weak Freese-Nation property.* S.F.) ❑

2 $P_{\mathcal{S}}$ and $B_{\mathcal{S}}$

In this section we introduce a construction of partial orderings $P_{\mathcal{S}}$ and Boolean algebras $B_{\mathcal{S}}$ which will be used in Sections 3 and 4. For $S \subseteq \kappa$ and an indexed family $\mathcal{S} = \langle S_\alpha : \alpha \in S \rangle$ of subsets of κ, let

$$P_{\mathcal{S}} = \{x_i : i \in \kappa\} \cup \{y_\alpha : \alpha \in S\}$$

where x_i's and y_α's are pairwise distinct, and let $\leq_{\mathcal{S}}$ be the partial ordering on $P_{\mathcal{S}}$ defined by

$$\begin{aligned} p \leq_{\mathcal{S}} q \ \Leftrightarrow \ & p = q \ \text{ or} \\ & p = x_i \text{ and } q = y_\alpha \text{ for some } i \in \kappa \text{ and } \alpha \in S \text{ with } i \in S_\alpha \ . \end{aligned}$$

Let $B_{\mathcal{S}}$ be the Boolean algebra generated freely from $P_{\mathcal{S}}$ except the relation $\leq_{\mathcal{S}}$. Note that the identity map on $P_{\mathcal{S}}$ canonically induces a strictly order-preserving embedding of $P_{\mathcal{S}}$ into $S_{\mathcal{S}}$.

Proposition 2.1 *Suppose that* $\mathrm{cf}(\kappa) \geq \omega_2$*,* $S \subseteq \kappa$ *is stationary such that* $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_1\}$ *and* $\mathcal{S} = \{S_\alpha : \alpha \in S\}$ *is such that* S_α *is a cofinal subset of* α *for each* $\alpha \in S$*. If* $P_{\mathcal{S}}$ *is embedded into a partial ordering* P *by a strictly order-preserving mapping*

then P *does not have the weak Freese-Nation property. In particular,* $B_{\mathcal{S}}$ *and its completion do not have the weak Freese-Nation property.*

Proof Without loss of generality, we may assume that $P_{\mathcal{S}}$ is a subordering of P. Assume to the contrary that there is a weak Freese-Nation mapping $f : P \to [P]^{\leq\aleph_0}$. Let

$$\begin{aligned} C = \{\xi < \kappa :& \forall \eta < \xi \, \forall p \in F(x_\eta) \\ & (\exists \alpha \in S \, (x_\eta \leq p \leq y_\alpha) \ \to \ \exists \alpha' \in S \cap \xi \, (x_\eta \leq p \leq y_{\alpha'}))\}. \end{aligned}$$

Then C is a club subset of κ. Let $\alpha \in C \cap S$ and let

$$A = \{p \in F(y_\alpha) : \exists \eta \in S_\alpha (p \in F(x_\eta) \ \wedge \ x_\eta \leq p \leq y_\alpha)\}.$$

Since $\alpha \in C$, for each $p \in A$ there is $\alpha_p < \alpha$ such that $p \leq y_{\alpha_p}$. Let $\alpha^* = \sup\{\alpha_p : p \in A\}$. Since A is countable we have $\alpha^* < \alpha$. Let $\beta \in S_\alpha \setminus \alpha^*$. Since $x_\beta \leq y_\alpha$, there is a $p \in A$ such that $x_\beta \leq p \leq y_\alpha$. Hence $x_\beta \leq y_{\alpha_p}$. But this is impossible since $\alpha_p \leq \beta$. S.F.) ❏ (Proposition 2.1)

3 Cohen models

Consider the following principle:

(**) There is a sequence $\langle S_\alpha : \alpha \in Lim(\omega_2)\rangle$ such that each S_α is a cofinal subset of α and that for any pairwise disjoint $\langle x_\beta : \beta < \omega_1\rangle$ with $x_\beta \in [\omega_2]^{<\aleph_0}$ for $\beta < \omega_1$, there are $\beta_0 < \beta_1 < \omega_1$ such that $x_{\beta_0} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_1} \cap Lim(\omega_2)$ and that $x_{\beta_1} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_0} \cap Lim(\omega_2)$.

Proposition 3.1 *Let* $P = \mathrm{Fn}(\omega_2, 2)$. *Then* $\Vdash_P$ "$(**)$".

Proof Without loss of generality we may assume $P = \mathrm{Fn}(\bigcup_{\alpha\in Lim(\omega_2)} \alpha \times \{\alpha\}, 2)$. For $\alpha \in Lim(\omega_2)$, let $\dot{S}_\alpha$ be a P-name such that $\Vdash_P$ "$\dot{S}_\alpha = \{\beta \in \alpha : \dot{g}(\beta, \alpha) = 1\}$" where $\dot{g}$ is the canonical name for the generic function. By genericity, $\Vdash_P$ "$\dot{S}_\alpha$ is cofinal in α" for every $\alpha \in Lim(\omega_2)$. Let $\dot{S}$ be a P-name such that $\Vdash_P$ "$\dot{S} = \langle \dot{S}_\alpha : \alpha \in Lim(\omega_1)\rangle$".

To show that $\dot{S}$ is forced to satisfy the property in $(**)$, let $\langle \dot{x}_\beta : \beta < \omega_1\rangle$ be a P-name of a sequence of pairwise disjoint finite subsets of ω_2. For each $\beta < \omega_1$, let p_β and $x_\beta \in [\omega_2]^{<\aleph_0}$ be such that $p_\beta \Vdash_P$ "$\dot{x}_\beta = x_\beta$". By thinning out the index set ω_1, we may assume without loss of generality that $\mathrm{dom}(p_\beta)$, $\beta < \omega_1$ form a Δ-system with the root d and $p_\beta \restriction d$, $\beta < \omega_1$ are all equal to the same $p \in P$. Since p_β, $\beta < \omega_1$ are then pairwise compatible, x_β, $\beta < \omega_1$ are pairwise disjoint. Further, we may assume also that s_β, $\beta < \omega_1$ form a Δ-system with the root s where $s_\beta = \{\gamma : (\gamma, \alpha) \in \mathrm{dom}(p_\beta)$ for some $\alpha < \omega_2\}$.

Let $\beta_0 < \beta_1 < \omega_1$ be such that $x_{\beta_0} \cap s = \emptyset$, $x_{\beta_1} \cap s = \emptyset$, $x_{\beta_0} \cap s_{\beta_1} = \emptyset$ and $x_{\beta_1} \cap s_{\beta_0} = \emptyset$. Let

$$\begin{aligned} p^* = p_{\beta_0} \cup p_{\beta_1} \cup \{((\beta, \alpha), 0) : \beta \in x_{\beta_0},\ \alpha \in x_{\beta_1} \cap Lim(\omega_2)\} \\ \cup \{((\beta, \alpha), 0) : \beta \in x_{\beta_1},\ \alpha \in x_{\beta_0} \cap Lim(\omega_2)\} \end{aligned}$$

Then $p^* \Vdash_P$ "$\dot{x}_{\beta_0} \cap \dot{S}_\alpha = \emptyset$" for all $\alpha \in \dot{x}_{\beta_1} \cap Lim(\omega_2)$ and $p^* \Vdash_P$ "$\dot{x}_{\beta_1} \cap \dot{S}_\alpha = \emptyset$" for all $\alpha \in \dot{x}_{\beta_0} \cap Lim(\omega_2)$. S.F.) ❑ (Proposition 3.1)

Proposition 3.2 *Suppose that* $\langle S_\alpha : \alpha \in Lim(\omega_2)\rangle$ *is as in* $(**)$. *Let* $S = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$ *and* $\mathcal{S} = \langle S_\alpha : \alpha \in S\rangle$. *Then* $B_{\mathcal{S}}$ *satisfies the c.c.c.*

Proof Otherwise we can find $I_\alpha \in [\omega_2]^{<\aleph_0}$, $J_\alpha \in [S]^{<\aleph_0}$ for $\alpha < \omega_1$ and $t(\alpha, i)$, $u(\alpha, \xi) \in \{+1, -1\}$ for each $i \in I_\alpha$, $\xi \in J_\alpha$ and $\alpha < \omega_1$ such that

$$z_\alpha = \prod_{i\in I_\alpha} t(\alpha, i)\, x_i \cdot \prod_{\xi\in J_\alpha} u(\alpha, \xi)\, y_\xi, \quad \alpha < \omega_1$$

form a pairwise disjoint family of elements of ${B_{\mathcal{S}}}^+$.

By Δ-system argument, we may assume that $I_\alpha \cup J_\alpha$, $\alpha < \omega_1$ are pairwise disjoint. Applying $(**)$ to $\langle I_\alpha \cup J_\alpha : \alpha < \omega_1 \rangle$, we find $\beta_0 < \beta_1 < \omega_1$ such that $I_{\beta_0} \cap S_\xi = \emptyset$ for all $\xi \in J_{\beta_1}$ and that $I_{\beta_1} \cap S_\xi = \emptyset$ for all $\xi \in J_{\beta_0}$. By definition of $B_{\mathcal{S}}$, it follows that $z_{\beta_0} \cdot z_{\beta_1} \neq 0$. This is a contradiction. S.F.) ❑ (Proposition 3.2)

Theorem 3.3 *In a Cohen model (i.e. any model obtained by adding $\geq \aleph_2$ Cohen reals) there is a c.c.c. complete Boolean algebra B of density $\aleph_2$ without the weak Freese-Nation property.*

Proof By Proposition 3.1, $(**)$ holds in a Cohen model. Hence $B_{\mathcal{S}}$ as in Proposition 3.2 satisfies the c.c.c. By Proposition 2.1, the completion of $B_{\mathcal{S}}$ does not have the weak Freese-Nation property. S.F.) ❑ (Theorem 3.3)

Corollary 3.4 *The weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ does not imply the weak Freese-Nation property of all c.c.c. complete Boolean algebras.*

Proof If we start from a model of CH and add $\aleph_2$ Cohen reals, then $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property in the resulting model (see e.g. [5]). On the other hand, by Theorem 3.3, there is a c.c.c. complete Boolean algebra without the weak Freese-Nation property in such a model. S.F.) ❑ (Corollary 3.4)

Under CH, every c.c.c. complete Boolean algebra of size $\aleph_2$ has the weak Freese-Nation property ([4]). Hence it follows from the result above that CH implies the negation of the principle $(**)$. This can be also seen directly as follows:

Proposition 3.5 CH *implies* $\neg(**)$.

Proof Let $\langle S_\alpha : \alpha \in \mathit{Lim}(\omega_2) \rangle$ be any sequence such that each S_α is a cofinal subset of α for $\alpha \in \mathit{Lim}(\omega_2)$. To show that $\langle S_\alpha : \alpha \in \mathit{Lim}(\omega_2) \rangle$ is not as in $(**)$, let χ be sufficiently large and let $M \prec \mathcal{H}(\chi)$ be such that $|M| = \aleph_1$; $\langle S_\alpha : \alpha \in \mathit{Lim}(\omega_2) \rangle \in M$; $\omega_1 \subseteq M$; $\omega_2 \cap M \in \omega_2$ and, letting $\gamma = \omega_2 \cap M$, $\mathrm{cf}(\gamma) = \omega_1$. By CH — and since $\omega_1 \subseteq M$ and $\mathrm{cf}(\gamma) = \omega_1$, $[\gamma]^{\aleph_0} \subseteq M$.

Now choose by induction distinct α^0_β, $\alpha^1_\beta < \gamma$ for $\beta < \omega_1$ such that (1) $\alpha^0_\beta \in S_\gamma$, and (2) $\{\alpha^0_\xi : \xi < \beta\} \subseteq S_{\alpha^1_\beta}$ for all $\beta < \omega_1$. (2) is possible: since $\{\alpha^0_\xi : \xi < \beta\} \subseteq S_\gamma$ and $\{\alpha^0_\xi : \xi < \beta\} \in M$, we have

$$M \models \exists \nu < \omega_2 (\sup\{\alpha^1_\xi : \xi < \beta\} < \nu \wedge \{\alpha^0_\xi : \xi < \beta\} \subseteq S_\nu).$$

Let $x_\beta = \{\alpha^0_\beta, \alpha^1_\beta\}$ for $\beta < \omega_1$. Then there are no $\beta_0 < \beta_1 < \omega_1$ such that $x_{\beta_0} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_1} \cap \mathit{Lim}(\omega_2)$. S.F.) ❑ (Proposition 3.5)

4 The Weak Freese-Nation property of c.c.c. complete Boolean algebras under GCH

In [5] it is proved that, assuming CH and a weak form of square principle at singular cardinals of cofinality ω, every c.c.c. complete Boolean algebra has the weak Freese-Nation property. In this section we show that even GCH does not suffice for this result.

Hajnal, Juhász and Shelah [9] showed that, starting from a model with a supercompact cardinal, a model of GCH and the following principle can be constructed:

(∗∗∗) There are a stationary $S \subseteq \{\alpha < \omega_{\omega+1} : \mathrm{cf}(\alpha) = \omega_1\}$ and a family $\mathcal{S} = \langle S_\alpha : \alpha \in S\rangle$ such that each S_α is a cofinal subset of α of ordertype ω_1 and that, for all distinct α, $\beta \in S$, $S_\alpha \cap S_\beta$ is finite.

Proposition 4.1 *Suppose that $\mathcal{S} = \langle S_\alpha : \alpha \in S\rangle$ is as in (∗∗∗). Then $B_{\mathcal{S}}$ satisfies the c.c.c..*

Proof Otherwise we can find $I_\alpha \in [\omega_{\omega+1}]^{<\aleph_0}$, $J_\alpha \in [S]^{<\aleph_0}$, $\alpha < \omega_1$ and $t(\alpha, i)$, $u(\alpha, \xi) \in \{+1, -1\}$ for each $\alpha < \omega_1$, $i \in I_\alpha$ and $\xi \in J_\alpha$ such that

$$z_\alpha = \prod_{i \in I_\alpha} t(\alpha, i)\, x_i \cdot \prod_{\xi \in J_\alpha} u(\alpha, \xi)\, y_\xi, \quad \alpha < \omega_1$$

form a pairwise disjoint family of elements of ${B_{\mathcal{S}}}^+$.

By Δ-system argument, we may assume that $I_\alpha \cup J_\alpha$, $\alpha < \omega_1$ are pairwise disjoint and each I_α has the same size, say n.

For $\alpha < \beta < \omega_1$, since $z_\alpha \cdot z_\beta = 0$, either (I) there is $\eta \in J_\alpha$ such that $I_\beta \cap S_\eta \neq \emptyset$ or else (II) there is $\xi \in J_\beta$ such that $I_\alpha \cap S_\xi \neq \emptyset$. If (I) holds then let us say that (α, β) is of type (I).

Now, one of the following two cases should hold. We show that both of them lead to a contradiction.

Case I. There is an infinite subset S of ω such that for every $\beta \in \omega_1 \setminus \omega$, $\{k \in S : (k, \beta)$ is of type (I)$\}$ is cofinite in S. In this case, by thinning out the index set ω_1, we may assume that, for any $k \in \omega$ and $\beta \in \omega_1 \setminus \omega$, (k, β) is of type (I). Since $|\, I_\alpha \,| = n$, for all $\beta \in \omega_1 \setminus \omega$, there are $0 \leq i^0(\beta) < i^1(\beta) < n + 1$ such that $I^*_\beta = I_\beta \cap S_{i^0(\beta)} \cap S_{i^1(\beta)} \neq \emptyset$ by Pigeonhole Principle. Hence we can find an infinite $X \subseteq \omega_1 \setminus \omega$ and $0 \leq i^0 < i^1 < n + 1$ such that $i^0(\beta) = i^0$ and $i^1(\beta) = i^1$ for all $\beta \in X$. But then $\bigcup_{\beta \in X} I^*_\beta \subseteq S_{i^0} \cap S_{i^1}$. Since the set on the left side is infinite as an infinite disjoint union of non-empty sets, this is a contradiction to (∗∗∗).

Case II. For any infinite subset $S \subseteq \omega$, there is $\beta \in \omega_1 \setminus \omega$ such that for infinitely many $k \in S$, (k, β) is not of type (I). In this case, by thinning out the

index set ω_1, we may assume that for each $k \in \omega$, there is $\xi(k) \in J_\omega$ such that $I_k \cap S_{\xi(k)} \neq \emptyset$. Note that J_ω is finite. So by thinning out further the first ω elements of the index set ω_1, we may assume that there is $\xi_0 \in J_\omega$ such that $I_k \cap S_{\xi_0} \neq \emptyset$ for every $k < \omega$. Similarly we may also assume that there are $\xi_i \in J_{\omega+i}$ for $1 \leq i \leq n$ such that $I_k \cap S_{\xi_i} \neq \emptyset$ for every $k < \omega$. For each $k < \omega$, we can find $i^0(k) < i^1(k) \leq n$ such that $I_k^* = I_k \cap S_{\xi_{i^0(k)}} \cap S_{\xi_{i^1(k)}} \neq \emptyset$ by Pigeonhole Principle. Since there are only $n(n-1)/2$ possibilities of $i^0(k) < i^1(k) \leq n$, there are $i^0 < i^1 \leq n$ and an infinite set $X \subseteq \omega$ such that for every $k \in X$, $i^0(k) = i^0$ and $i^1(k) = i^1$. It follows that $S_{\xi_{i^0}} \cap S_{\xi_{i^1}} \supseteq \bigcup_{k \in X} I_k^*$. Since $S_{\xi_{i^0}} \cap S_{\xi_{i^1}}$ is finite, this is a contradiction.

S.F.) ❑ (Proposition 4.1)

Theorem 4.2 *It is consistent with* GCH *(modulo the consistency strength of a supercompact cardinal) that there is a c.c.c. complete Boolean algebra without the weak Freese-Nation property.*

Proof Let $\mathcal{S}$ be a family as in $(***)$. By Proposition 4.1 and Proposition 2.1 the completion of $B_{\mathcal{S}}$ is a c.c.c. complete Boolean algebra without the weak Freese-Nation property.

S.F.) ❑ (Theorem 4.2)

In [5] it is proved that under CH and a very weak version of the square principle at $\aleph_\omega$, every c.c.c. complete Boolean algebra of cardinality $\aleph_{\omega+1}$ has the weak Freese-Nation property. Hence we see that consistency strength of some large cardinal is involved in $(***)$. This can be also seen directly as follows.

First let us review the weak form of the square principle used in [5]. $\square^{***}_{\aleph_1,\mu}$ is the following assertion: there exists a sequence $\langle C_\alpha : \alpha < \mu^+\rangle$ and a club set $D \subseteq \mu^+$ such that for $\alpha \in D$ with $\mathrm{cf}(\alpha) \geq \omega_1$

(y1) $C_\alpha \subseteq \alpha$, C_α is unbounded in α;

(y2) $[\alpha]^{<\omega_1} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{<\omega_1}$ (with respect to $\subseteq$).

It can be easily seen that $\square^{***}_{\aleph_1,\mu}$ follows from the very weak square principle for μ by Foreman and Magidor [3] (see [5]).

Proposition 4.3 $2^{\aleph_0} < \aleph_\omega$ *and* $\square^{***}_{\aleph_1,\omega_\omega}$ *implies the negation of* $(***)$.

Proof Let $\langle C_\alpha : \alpha < \omega_\omega+\rangle$ and $D \subseteq {\omega_\omega}^+$ be as in the definition of $\square^{***}_{\aleph_1,\omega_\omega}$. Suppose that S and $\langle S_\alpha : \alpha \in S\rangle$ are as in $(***)$. We show that there are $\xi, \eta \in S$, $\xi \neq \eta$ such that $S_\xi \cap S_\eta$ is not finite. By replacing S by $S \cap D$ and C_α by $C_\alpha \cap S_\alpha$ for $\alpha \in S \cap D$, we may assume that $S \subseteq D$ and $\langle S_\alpha : \alpha \in S\rangle$ is just the sequence $\langle C_\alpha : \alpha \in S\rangle$.

By (y2), for each $\alpha \in S$, there is $\beta_\alpha < \alpha$ of countable cofinality such that $C_\alpha \cap C_{\beta_\alpha}$ is infinite. By Fodor's theorem we may assume that every β_α, $\alpha \in S$ are

the same β. Since there are only $< \aleph_\omega$ different subsets of the countable set C_β, we can find ξ, $\eta \in S$, $\xi \neq \eta$ such that $S_\xi \cap C_\beta = S_\eta \cap C_\beta$. It follows that $S_\xi \cap S_\eta$ is infinite. S.F.) ❑ (Proposition 4.3)

5 The weak Freese-Nation property of c.c.c. complete Boolean algebras under weak square principles

In this section, we investigate the c.c.c. complete Boolean algebras B for which we can prove (in ZFC or in some extension of it) that the weak Freese-Nation property of B is equivalent with the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$. Lemma 1.1 was an easy observation in this direction. S. Koppelberg observed in [10] that the Cohen algebra $\mathbb{C}(\omega)$ is such a Boolean algebra.

Since $(\mathcal{P}(\omega), \subseteq)$ can be embedded in every complete Boolean algebra, it follows from Proposition 1.2 (a) and one of the remarks before it that, if $(\mathcal{P}(\omega), \subseteq)$ does not have the weak Freese-Nation property then no complete Boolean algebra can have the weak Freese-Nation property.

Proposition 5.1 *The measure algebra $\mathbb{R}(\omega)$ has the weak Freese-Nation property if and only if $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property.*

Proof By the remark above and by Proposition 1.2,(f) together with one of the remarks before Proposition 1.2, it is enough to find a strictly order-preserving embedding of $\mathbb{R}(\omega)$ into $({}^{\omega}\mathbb{R}, \leq)$. We may replace ω by the countable set $Clop({}^{\omega}2)$ where $Clop({}^{\omega}2)$ denotes the clopen sets of the Cantor space ${}^{\omega}2$.

We define $e : \mathbb{R}(\omega) \to {}^{Clop({}^{\omega}2)}\mathbb{R}$ by taking $e(a)(c) = \mu(a \cap c)$ for $c \in Clop({}^{\omega}2)$. Then clearly e is a order-preserving map of $\mathbb{R}(\omega)$ into ${}^{Clop({}^{\omega}2)}\mathbb{R}$.

To show that e is a strictly order-preserving, assume that $\mu(a \setminus b) > 0$. Then there is $c \in Clop({}^{\omega}2)$ such that $\mu((a \setminus b) \cap c) > \mu(c)/2$. Then $e(a)(c) > \mu(c)/2 > e(b)(c)$, so $e(a) \not\leq e(b)$. S.F.) ❑ (Proposition 5.1)

In a similar way, we can also show that $\mathbb{R}(\omega)$ is a retract of $\mathcal{P}(\omega)$ as a partially ordered set though it is known that $\mathbb{R}(\omega)$ is *not* a retract of $\mathcal{P}(\omega)$ as a Boolean algebra: the mapping $e : \mathbb{R}(\omega) \to \mathcal{P}(Clop({}^{\omega}2) \times \mathbb{Q})$ defined by $e(c) = \{(a, q) : a \in Clop({}^{\omega}2),\ q < \mu(a \cap c)\}$ is easily seen to be a strictly order preserving embedding.

In general, the weak Freese-Nation property of $\mathbb{C}(\kappa)$ or that of $\mathbb{R}(\kappa)$ for arbitrary $\kappa \geq \omega$ is not equivalent with the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$. In the next section we shall give a model where $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property while $\mathbb{C}(\aleph_\omega)$ (and hence also $\mathbb{R}(\aleph_\omega)$) does not.

However the equivalence does hold if $\kappa < \aleph_\omega$ or some consequences of $\neg 0^{\#}$ are available. To prove this, we need the following instance of Theorem 7 in [5]:

Theorem 5.2 (Theorem 7 in [5] for $\kappa = \aleph_1$) *Suppose that $\mu > \mathrm{cf}(\mu) = \omega$, $\mathrm{cf}([\lambda]^{\aleph_0}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square^{***}_{\aleph_1,\mu}$ holds. Then for any sufficiently large regular χ and $x \in \mathcal{H}(\chi)$, there is a matrix $(M_{\alpha,i})_{\alpha<\mu^+, i<\omega}$ such that*

(1) *$M_{\alpha,i} \prec \mathcal{H}(\chi)$, $x \in M_{\alpha,\beta}$, $\omega_1 \subseteq M_{\alpha,i}$ and $|\, M_{\alpha,i} \,| < \mu$ for all $\alpha < \mu^+$ and $i < \omega$;*

(2) *$(M_{\alpha,i})_{i<\omega}$ is an increasing sequence for each $\alpha < \mu^+$;*

(3) *If $\alpha < \mu^+$ and $\mathrm{cf}(\alpha) \geq \omega_1$, then there is an $i^* < \omega$ such that for every $i^* \leq i < \omega$, $[M_{\alpha,i}]^{\aleph_0} \cap M_{\alpha,i}$ is cofinal in $([M_{\alpha,i}]^{\aleph_0}, \subseteq)$;*

(4) *Let $M_\alpha = \bigcup_{i<\omega} M_{\alpha,i}$ for $\alpha < \mu^+$. Then $M_\alpha \prec \mathcal{H}(\chi)$ (by (1) and (2)). Moreover $(M_\alpha)_{\alpha<\mu^+}$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$.* S.F.) ❏

For a complete Boolean algebra B and $X \subseteq B$, let us denote by $\langle X \rangle^{\mathrm{cm}}_B$ the complete subalgebra of B generated completely by X.

Theorem 5.3 *Let λ be a cardinal. Suppose that for every $\mu < \lambda$ with $\mu > \mathrm{cf}(\mu) = \omega$, we have:*

(i) $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$*;*

(ii) $\square^{***}_{\aleph_1,\mu}$.

Then for any c.c.c. complete Boolean algebra B with a complete generator of size $< \lambda$, B has the weak Freese-Nation property if and only if every complete subalgebra of B generated completely by a countable subset of B has the weak Freese-Nation property.

Proof "Only if" part of the theorem follows from Proposition 1.2 (b). "If" part of the theorem is proved by induction on the minimal cardinality of a subset X of B completely generating B.

If X is countable, then there is nothing to prove. Let $\mu = |\, X \,| < \lambda$ and suppose that we have the theorem for any c.c.c. complete Boolean algebra with a complete generator of size $< \mu$.

If $\mathrm{cf}(\mu) > \omega$, then letting $X = \{x_\alpha : \alpha < \mu\}$ and $B_\beta = \langle \{x_\alpha : \alpha < \beta\} \rangle^{\mathrm{cm}}_B$ for $\beta < \mu$, we have $B_\beta \leq_c B$ and hence $B_\beta \leq_\sigma B$ for all $\beta < \mu$. By induction hypothesis, every B_β, $\beta < \mu$ has the weak Freese-Nation property. By the c.c.c. of B $B_\gamma = \bigcup_{\beta<\gamma} B_\beta$ for all limit $\gamma < \mu$ with $\mathrm{cf}(\gamma) > \omega$ and also $B = \bigcup_{\beta<\mu} B_\beta$. Hence by Proposition 1.2 (c), it follows that B has the weak Freese-Nation property.

If $\mathrm{cf}(\mu) = \omega$, then there is $(M_{\alpha,i})_{\alpha<\mu^+, i<\omega}$ as in the previous theorem for $x = (B, X)$.

For $\alpha < \mu^+$ and $i < \omega$, let $B_{\alpha,i} = \langle B \cap M_{\alpha,i} \rangle^{\mathrm{cm}}_B$ and $B_\alpha = \bigcup_{i<\omega} B_{\alpha,i}$.

Claim 5.3.1 *For every $\alpha < \mu^+$, B_α has the weak Freese-Nation property and $B_\alpha \leq_\sigma B$.*

$\vdash$ For every $i < \omega$, $B_{\alpha,i}$ has the weak Freese-Nation property by induction hypothesis. Since $B_{\alpha,i} \leq_c B$ for every $i < \omega$ it follows that $B_\alpha \leq_\sigma B$. Also by Proposition 1.2 (c) it follows that B_α has the weak Freese-Nation property. $\dashv$ (Claim 5.3.1)

Claim 5.3.2 *If $\gamma < \mu^+$ and $\mathrm{cf}(\gamma) > \omega$, then $B_\gamma = \bigcup_{\alpha<\gamma} B_\gamma$.*

$\vdash$ Suppose that $a \in B_\gamma$. Then, by the c.c.c. of B, there is an $i < \omega$ and $s \in [B \cap M_{\gamma,i}]^{\aleph_0}$ such that $a \in \langle s\rangle_B^{\mathrm{cm}}$. By (3) in Theorem 5.2, we may assume that $s \in M_{\gamma,i}$. By (4), there is $\alpha < \gamma$ and $j < \omega$ such that $s \in M_{\alpha,j}$. It follows that $s \subseteq M_{\alpha,j}$ and hence $a \in B_{\alpha,j} \subseteq B_\alpha$. $\dashv$ (Claim 5.3.2)

Claim 5.3.3 $B = \bigcup_{\alpha<\mu^+} B_\alpha$.

$\vdash$ By the last statement of (4) in Theorem 5.2 and (i), $[X]^{\aleph_0} \cap \bigcup_{\alpha<\mu^+} M_\alpha$ is cofinal in $([X]^{\aleph_0}, \subseteq)$.

Suppose now that $a \in B$. Then by the c.c.c. of B, there is a countable $s \in [X]^{\aleph_0}$ such that $a \in \langle s\rangle_B^{\mathrm{cm}}$. By the remark above, we may assume that $s \in \bigcup_{\alpha<\mu^+} M_\alpha$, say $s \in M_{\alpha^*,i^*}$ for some $\alpha^* < \mu^+$ and $i^* < \omega$. Then $s \subseteq B \cap M_{\alpha^*,i^*}$ and hence $a \in B_{\alpha^*,i^*} \subseteq B_{\alpha^*}$. $\dashv$ (Claim 5.3.3)

Now by Theorem 1.2 (c) and the claims above, it follows that B has the weak Freese-Nation property. S.F.) ❏ (Theorem 5.3)

Corollary 5.4 *Let λ be as in Theorem 5.3 and $\omega \leq \kappa < \lambda$. Then the following are equivalent:*

(0) $(\mathcal{P}(\omega), \subseteq)$ *has the weak Freese-Nation property;*

(1) $\mathbb{C}(\kappa)$ *has the weak Freese-Nation property;*

(2) $\mathbb{R}(\kappa)$ *has the weak Freese-Nation property.*

Proof (1) $\Rightarrow$ (0) and (2) $\Rightarrow$ (0) follows from Proposition 1.2 (b).

For (0) $\Rightarrow$ (2) assume that $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property. Since $\mathbb{R}(\kappa)$ has the complete generator $Clop({}^\kappa 2)$, it is enough by Theorem 5.3 to show that every subalgebra of $\mathbb{R}(\kappa)$ has the weak Freese-Nation property. Let A be such an algebra then there is $X \subseteq [\kappa]^{\aleph_0}$ such that $A \leq_c \mathbb{R}(X)$. Since $\mathbb{R}(X)$ has the weak Freese-Nation property by Proposition 5.1, it follows by Proposition 1.2 (b) that A also has the weak Freese-Nation property. S.F.) ❏ (Corollary 5.4)

Note that the conditions on λ in Theorem 5.3 hold vacuously for $\lambda = \aleph_\omega$. Hence we obtain the following as a special case of the corollary above:

Corollary 5.5 *The following are equivalent (in* ZFC*):*

(0) $(\mathcal{P}(\omega), \subseteq)$ *has the weak Freese-Nation property;*

(1) $\mathbb{C}(\aleph_n)$ *has the weak Freese-Nation property for some/all* $n \in \omega$*;*

(2) $\mathbb{R}(\aleph_n)$ *has the weak Freese-Nation property for some/all* $n \in \omega$. S.F.) ❑

6 Chang's Conjecture

In this section, we give a negative answer to Problem 3 mentioned in the introduction (see Theorem 6.1 below) and show that Corollary 5.5 in the last section is the optimal assertion among what we can obtain in ZFC.

Theorem 6.1 *Suppose that* V_0 *is a transitive model of* ZFC *such that*

$$V_0 \models \mathrm{GCH} + (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)\,.$$

Let P *be a c.c.c. partial ordering in* V_0 *of cardinality* $\aleph_1$ *adding a dominating real. Let* $\eta \in {}^\omega\omega$ *be a dominating real over* V_0 *generically added by* P *and let* $V_1 = V_0[\eta]$. *Note that* $V_1 \models$ GCH. *In* V_1 *let* $Q = \mathrm{Fn}(\aleph_\omega, \omega)$ *and let* $\dot{Q}$ *be a corresponding* P*-name. Then we have:*

$$V_1 \models \;\Vdash_Q \text{“}(\mathcal{P}(\omega), \subseteq) \textit{ does not have the weak Freese-Nation property”}.$$

Proof In the proofs of this and next theorems, we shall denote by a dotted symbol a name of an element in a generic extension. By the same symbol without the dot, we denote the corresponding element in a fixed generic extension. Without further mention, we shall identify $P * \dot{Q}$ names with the corresponding Q-name in V_1 and vice versa.

Now toward a contradiction, assume that there is a Q-name $\dot{F}$ in V_1 such that

$$(\otimes) \qquad V_1 \models \;\Vdash_Q \text{“}\dot{F} \text{ is a weak Freese-Nation mapping over } ({}^\omega\omega, \leq^*)\text{”}.$$

Let $\dot{\varphi}$ be a $P*\dot{Q}$-name of the function $\aleph_\omega \to \omega$ generically added by Q over V_1. Let $V_2 = V_1[\varphi]$. By GCH, we can find in V_0 a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ in $(\prod_{n\in\omega} \aleph_n, \leq^*)$. Without loss of generality, we may assume that for every $\alpha < \aleph_{\omega+1}$ and $n \in \omega$, $f_\alpha(n) \in \aleph_n \setminus \aleph_{n-1}$ (where we set $\aleph_{-1} = 0$). For each $\alpha \in \aleph_{\omega+1}$, let

$$g_\alpha = \varphi \circ f_\alpha\,.$$

Let χ be sufficiently large and let $N \prec (\mathcal{H}(\chi), \in)$ be such that N contains every thing we need in the course of the proof, $|\,\aleph_\omega \cap N\,| = \aleph_0$ and $otp(\aleph_{\omega+1} \cap N) = \omega_1$ — the latter two conditions are possible by $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

In V_0, let $\{\xi_{n,k} : k \in \omega\}$ be an enumeration of $(\aleph_n \setminus \aleph_{n-1}) \cap N$ for each $n \in \omega$. Here again, we use the convention that $\aleph_{-1} = 0$. Let $\dot{h}^*$ be a $P * \dot{Q}$-name of an element of ${}^\omega\omega$ such that

$$\Vdash_{P*\dot{Q}} \text{“}\dot{h}^*(n) = \max\{\dot{\varphi}(\xi_{n,k}) : k < \dot{\eta}(n)\} \text{ for all } n \in \omega\text{”}.$$

Claim 6.1.1 *For every $\alpha \in \aleph_{\omega+1} \cap N$, $\Vdash_{P*\dot{Q}}$ “$\dot{g}_\alpha \leq^* \dot{h}^*$”.*

$\vdash$ Since $\alpha \in N$ we have $f_\alpha \in N$. Hence there is a function $e_\alpha : \omega \to \omega$ in V_0 such that $f_\alpha(n) = \xi_{n,e_\alpha(n)}$. Since η is dominating, there is $n^* \in \omega$ such that

$$V_1 \models e_\alpha \restriction \omega \setminus n^* \leq \eta \restriction \omega \setminus n^*.$$

By definition of $\dot{h}^*$, it follows that

$$V_2 \models g_\alpha(n) = \varphi \circ f_\alpha(n) = \varphi(\xi_{n,e_\alpha(n)}) \leq h^*(n)$$

for all $n \geq n^*$. $\dashv$ (Claim 6.1.1)

Let $N_0 = N$, $N_1 = N_0[\eta]$ and $N_2 = N_1[\varphi]$. Then we have $N_2 \prec \mathcal{H}(\chi)[\eta][\varphi]$.

Let $\dot{h}_n \in N_0$, $n \in \omega$ be $P * \dot{Q}$-names such that

$$\Vdash_{P*\dot{Q}} \text{“}\{\dot{h}_n : n \in \omega\} = \dot{F}(\dot{h}^*) \cap \dot{N}_2\text{”}.$$

Let $S_n \in [\aleph_\omega]^{\aleph_0} \cap N_0$ be such that, regarding $\dot{h}_n$ as a P-name of $\dot{Q}$-name,

$$V_1 \models \Vdash_P \text{“}\dot{h}_n \text{ is a } \mathrm{Fn}(S_n, \omega)\text{-name”}.$$

This is possible since P has the c.c.c. and $\Vdash_P$ “$\dot{Q}$ has the c.c.c.”. For each $n \in \omega$, let $s_n \in \prod_{n\in\omega} \aleph_n \cap N_0$ be defined (in N_0) by $s_n(k) = \sup S_n \cap \aleph_k$ for $k \in \omega$. Since $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ was taken to be a scale on $\prod_{n\in\omega} \aleph_n$, for each $n \in \omega$ there is $\alpha_n \in \aleph_{\omega+1} \cap N_0$ such that $s_n \leq^* f_{\alpha_n}$. Let $\alpha^* \in \aleph_{\omega+1} \cap N_0$ be such that $\sup\{\alpha_n : n \in \omega\} \leq \alpha^*$.

Now, by the choice of $\dot{h}_n$, $n \in \omega$, the following claim contradicts to $(\otimes)$ and Claim 6.1.1, and hence proves the theorem:

Claim 6.1.2 *$V_1 \models \Vdash_Q$ “$\dot{g}_\alpha \not\leq^* \dot{h}_n$ for all $n \in \omega$”.*

$\vdash$ Assume to the contrary that, in V_1, we have

$$q \Vdash_Q \text{“}\dot{g}_\alpha \restriction (\omega \setminus k) \leq \dot{h}_n \restriction (\omega \setminus k)\text{”}$$

for some $q \in Q$ and n, $k \in \omega$. We may assume that

$$s_n \restriction \omega \setminus k < f_{\alpha^*} \restriction \omega \setminus k$$

and $\sup(\mathrm{dom}(p)) \leq f_\alpha(m)$ for all $m \in \omega \setminus k$ as well. Working further in V_1, let $m^* = k+1$ and let $q' \leq q$ be such that

$$q' \Vdash_Q \text{“} \dot{h}_n(m^*) = j^* \text{”}$$

for some $j^* \in \omega$. Then $q'' = q \cup (q' \restriction S_n)$ also forces the same statement. Since $f_{\alpha^*}(m^*) \in \aleph_{m^*} \setminus (s_n(m^*) \cup \mathrm{dom}(p))$, we have

$$f^*_\alpha(m^*) \notin \mathrm{dom}(q'').$$

Hence

$$q^* = q'' \cup \{(f_{\alpha^*}(m^*), j^*+1)\}$$

is an element of Q and $q^* \leq q'' \leq q$. But

$$q^* \Vdash_Q \text{“} \dot{g}_{\alpha^*}(m^*) = \dot{\varphi} \circ f_{\alpha^*}(m^*) = j^*+1 > j^* = \dot{h}_n(m^*). \text{”}$$

This is a contradiction. $\dashv$ (Claim 6.1.2)

S.F.) ❑ (Theorem 6.1)

Theorem 6.2 *Suppose that V_0, P, η, $\dot{\eta}$, V_1 are as in the previous theorem. Then*

$$V_1 \models \mathbb{C}(\aleph_\omega) \textit{ does not have the weak Freese-Nation property.}$$

Proof Suppose to the contrary that $F \in V_1$ is a weak Freese-Nation mapping on $\mathbb{C}(\aleph_\omega)$. Let $X = \{x_\xi : \xi < \aleph_\omega\}$ be a free subset of $\mathbb{C}(\aleph_\omega)$ completely generating the whole $\mathbb{C}(\aleph_\omega)$. We may take $X \in V_0$.

Let $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be as in the proof of the previous theorem. For $n \in \omega$ let $c_n = x_n \cdot -\sum_{m<n} x_m$. $\{c_n : n \in \omega\}$ is then a partition of $\mathbb{C}(\aleph_\omega)$. For $\alpha < \aleph_{\omega+1}$ let

$$b_{\alpha,n} = \sum\nolimits_{m>n}(x_{f_\alpha(m)} \cdot c_m)$$

Let χ be sufficiently large and let $N \prec (\mathcal{H}(\chi), \in)$ be such that N contains every thing we need in the course of the proof, $|\,\aleph_\omega \cap N\,| = \aleph_0$ and $otp(\aleph_{\omega+1} \cap N) = \omega_1$. In V_0, let $\{\xi_{n,k} : k \in \omega\}$ be an enumeration of $(\aleph_n \setminus \aleph_{n-1}) \cap N$ for each $n \in \omega$. In $N_1 = N[\eta]$ let

$$b^* = \sum\nolimits_{n\in\omega}(c_n \cdot \sum\nolimits_{l<\eta(n)} x_{\xi_{n,l}})\,.$$

Claim 6.2.1 *For every $\alpha \in \aleph_{\omega+1} \cap N$, there is $n_\alpha \in \omega$ such that $b_{\alpha,n_\alpha} \leq b^*$.*

$\vdash$ In V_0, let $s_\alpha \in {}^\omega\omega$ be such that $f_\alpha(n) = \xi_{n,s_\alpha(n)}$ for all $n \in \omega$. Since η is dominating, there is an $n_\alpha \in \omega$ such that $s_\alpha \restriction \omega \setminus n_\alpha \leq \eta \restriction \omega \setminus n_\alpha$. By definition of $b_{\alpha,n}$ it follows that $b_{\alpha,n_\alpha} \leq b^*$. $\dashv$ (Claim 6.2.1)

Now, let $\langle d_n : n \in \omega \rangle$ be an enumeration of $F(b^*) \cap \mathbb{C}(\aleph_\omega) \restriction b^* \cap N$ and $\dot{d}_n$, $n \in \omega$ be corresponding P-names. We can choose these names so that $\langle \dot{d}_n : n \in \omega \rangle \subseteq N$.

For each n, there is $S_n \in [\aleph_{\omega+1}]^{\aleph_0} \cap N$ such that

$$\Vdash_P \text{“} \dot{d}_n \in \langle S_n \rangle^{\mathrm{cm}}_{\mathbb{C}(\aleph_\omega)} \text{”}.$$

Let $f^*_n \in \prod_{n\in\omega} \aleph_n \cap N$ be defined by

$$f^*_n(m) = \min(\aleph_m \setminus S_n)$$

for $m \in \omega$. Since $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ is a scale on $(\prod_{n\in\omega} \aleph_n, \leq^*)$, there is $\alpha^*_n \in \aleph_{\omega+1} \cap N$ such that $f^*_n \leq^* f_{\alpha^*_n}$. Let $\alpha^* \in \aleph_{\omega+1} \cap N$ be such that $\alpha^*_n < \alpha^*$ for all $n \in \omega$.

Then similarly to the proof of the previous theorem. The following claim gives a desired contradiction:

Claim 6.2.2 *For every $n \in \omega$, $b_{\alpha^*, n_{\alpha^*}} \not\leq d_n$.*

$\vdash$ For $n \in \omega$, let $m \in \omega$ be such that $n_{\alpha^*} < m$, $c_m \cdot -d_n \neq 0$ and $f^*_n(m) < f_{\alpha^*}(m)$. Then $f_{\alpha^*}(m) \notin S_n$. Hence $x_{f_{\alpha^*}(m)} \cdot c_m \not\leq d_n$ but $x_{f_{\alpha^*}(m)} \cdot c_m \leq b_{\alpha^*, n_{\alpha^*}}$ by definition of $b_{\alpha^*, n_{\alpha^*}}$. $\dashv$ (Claim 6.2.2)

S.F.) ❏ (Theorem 6.2)

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